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On a theorem of Garuti

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ABSTRACT

In this note we prove a refined version of the main theorem proved by Garuti (1996) in [2] on liftings of Galois covers between smooth curves. We also describe the structure of a certain pro-p quotient of the geometric Galois group of a p-adic open disc.

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0. Introduction

In what follows, R is a complete discrete valuation ring of unequal characteristic, $K \overset{\text{def}}{=} \mathrm{Fr}(R)$ the quotient field of R, $\mathrm{char}(K) = 0$ and k the residue field of R which we assume to be algebraically closed of characteristic $p > 0$. Let X be a smooth proper R-curve with special fibre $X_k \overset{\text{def}}{=} X \times_R k$ and $Y_k \to X_k$ a Galois (possibly ramified) cover between smooth and proper k-curves with Galois group G. In [2], Garuti proved the following theorem in connection with the problem of lifting Galois covers between smooth curves.

Theorem A (*Garuti*)**.** *There exists a finite extension R'/R and a finite Galois cover $f' : Y' \to X' \overset{\text{def}}{=} X \times_R R'$ with Galois group G, where Y' is a normal R'-curve (which in general need not be smooth over R'), and the natural morphism $f'_k : Y'_k \overset{\text{def}}{=} Y' \times_{R'} k \to X_k$ between special fibres is generically Galois with Galois group G. Moreover, there exists a factorisation $f_k : Y_k \xrightarrow{\nu} Y'_k \xrightarrow{f'_k} X_k$ where the morphism $\nu : Y_k \to Y'_k$ is a morphism of normalisation which is an isomorphism outside the ramified points and Y'_k is unibranch.*

We call f' as in Theorem A a Garuti lifting of the Galois cover f_k defined over R'. In this note we prove the following refined version of Theorem A.

Theorem B (*cf. Theorem 2.5.3*)**.** *We use the same notations as in Theorem A. Let H be a quotient of G and $g_k : Z_k \to X_k$ the corresponding Galois sub-cover of f_k with Galois group H. Let $h' : Z' \to X' \overset{\text{def}}{=} X \times_R R'$ be a Garuti lifting of the Galois cover h_k defined over the finite extension R'/R. Then there exists a finite extension R''/R' and a Garuti lifting $f'' : Y'' \to X'' \overset{\text{def}}{=} X \times_R R''$ of the Galois cover f_k over R'' which dominates h', i.e. we have a factorisation $f'' : Y'' \xrightarrow{g''} Z'' \overset{\text{def}}{=} Z' \times_{R'} R'' \xrightarrow{h'' \overset{\text{def}}{=} h' \times_{R'} R''} X''$ where $g'' : Y'' \to Z''$ is a finite morphism between normal R''-curves.*

In the course of proving Theorem B we prove a structure theorem concerning a certain quotient of the "geometric Galois group" of a p-adic open disc, which is the most relevant to the problem of lifting Galois covers between smooth curves. This result might be of interest independently from the lifting problem as investigated in [2]. Let $\tilde{X} \overset{\text{def}}{=} \mathrm{Spf}\, R[[T]]$, $\tilde{X}_K \overset{\text{def}}{=} \mathrm{Spec}(R[[T]] \times_R K)$ (a p-adic open disc over K) and $\mathcal{X} \overset{\text{def}}{=} \mathrm{Spf}\, R[[T]]\{T^{-1}\}$ the formal boundary of $\tilde{X}$ (cf. Section 2). Let Δ (resp. Δ') be the maximal pro-p group which classifies geometric Galois covers of $\tilde{X}$ (resp. of $\mathcal{X}$) which are pro-p and which are generically étale at the level of special fibres (cf. 2.3 and 2.4 for more precise definitions).

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Theorem C (*cf. Theorems 2.3.1 and 2.4.1*)**.** *The profinite group Δ is a free pro-p group. Moreover, there exists a natural morphism $\Delta' \to \Delta$ which makes Δ' into a direct factor of Δ (cf. 1.1 for the definition and characterisation of a direct factor of a free pro-p group).*

In Section 1 we collect some background material which is used in this note. This includes background material on pro-p groups and formal patching techniques. In Section 2 we prove Theorem B and our main result Theorem C concerning the structure of a certain quotient of the geometric Galois group of a p-adic open disc. Our approach in proving Theorems B and C is very close to the approach of Garuti in proving Theorem A. Our proof of Theorem B follows from Theorem C. The main ingredients we use to prove Theorem C are: (1) formal patching techniques à la Harbater (cf. Proposition 1.2.2), (2) a result of Garuti on the geometric étale pro-p fundamental group of a p-adic annulus of thickness zero (cf. Proposition 2.2.3) and (3) a result of Harbater, Katz and Gabber on the approximation of Galois extensions of local fields by Galois extensions of function fields.

1. Background

In this section we collect some background material which is used in this paper.

1.1. Complements on pro-p groups

In this sub-section we fix a prime integer $p > 1$. We recall some well-known facts on profinite pro-p groups that will be used in Section 2. First, we recall the following characterisations of free pro-p groups.

Proposition 1.1.1. *Let G be a profinite pro-p group. Consider the following properties:*

(i) *G is a free pro-p group.*
(ii) *The p-cohomological dimension of G satisfies* $\mathrm{cd}_p(G) \le 1$.
(iii) *Given a surjective homomorphism $\sigma : Q \twoheadrightarrow P$ between finite p-groups, and a continuous surjective homomorphism $\phi : G \twoheadrightarrow P$, there exists a continuous homomorphism $\psi : G \to Q$ such that the following diagram is commutative:*

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}"] \arrow[d, "\psi"'] & G \arrow[d, "\phi"'] \\ Q \arrow[r, "\sigma"] & P \end{tikzcd}$$

Then the following equivalences hold:

$$\text{(i)} \iff \text{(ii)} \iff \text{(iii)}.$$

Proof. Well-known (cf. [8], and [6], Theorem 7.7.4). □

Next, we recall the notion of a direct factor of a free pro-p group (cf. [2], 1, the discussion preceding Proposition 1.8).

Definition 1.1.2 (*Direct Factors of Free pro-p Groups*)**.** Let F be a free pro-p group and H a closed subgroup of F. Let $\iota : H \to F$ be the natural homomorphism. We say that H is a direct factor of F if there exists a continuous homomorphism $s : F \to H$ such that $s \circ \iota = \mathrm{id}_H$ (s is necessarily surjective). We then have a natural exact sequence

$$1 \to N \to F \overset{s}{\to} H \to 1,$$

where $N \overset{\mathrm{def}}{=} \mathrm{Ker}\, s$ and F is isomorphic to the free direct product

$$H * N.$$

In particular, N is also a direct factor of F (cf. loc. cit).

One has the following cohomological characterisation of direct factors of free pro-p groups.

Proposition 1.1.3. *Let H be a pro-p group and F a free pro-p group. Let $\sigma : H \to F$ be a continuous homomorphism. Assume that the map induced by σ on cohomology:*

$$h^1(\sigma) : H^1(F, \mathbb{Z}/p\mathbb{Z}) \to H^1(H, \mathbb{Z}/p\mathbb{Z})$$

is surjective, where $\mathbb{Z}/p\mathbb{Z}$ is considered as a trivial discrete module. Then H is a direct factor of F.

Proof. cf. [2], Proposition 1.8. □

1.2. Formal patching

In this sub-section we explain the procedure which allows to construct (Galois) covers of curves in the setting of formal geometry by patching together covers of formal affine curves with covers of formal fibres at closed points of the special fibre (cf. [7], 1, for more details). We also recall the (well-known) local-global principle for liftings of Galois covers of curves.

Let R be a complete discrete valuation ring with fraction field K, residue field k, and uniformiser π. Let X be an admissible formal R-scheme which is an R-curve, by which we mean that the special fibre $X_k \overset{\mathrm{def}}{=} X \times_R k$ is a reduced one-dimensional scheme of finite type over k. Let Z be a finite set of closed points of X. For a point $x \in Z$ let $X_x \stackrel{\text{def}}{=} \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ be the formal completion of X at x, which is the formal fibre at the point x. Let X' be a formal open sub-scheme of X whose special fibre is $X_k \setminus Z$. For each closed point $x \in Z$ let $\{\mathcal{P}_i\}_{i=1}^n$ be the set of minimal prime ideals of $\hat{\mathcal{O}}_{X,x}$ which contain π; they correspond to the branches $\{\eta_i\}_{i=1}^n$ of the completion of X_k at x, and let $X_{x,i} \stackrel{\text{def}}{=} \operatorname{Spf} \hat{\mathcal{O}}_{x,\mathcal{P}_i}$ be the formal completion of the localisation of X_x at $\mathcal{P}_i$. The local ring $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ is a complete discrete valuation ring. The set $\{X_{x,i}\}_{i=1}^n$ is the set of boundaries of the formal fibre X_x. For each $i \in \{1, \ldots, n\}$ we have a canonical morphism $X_{x,i} \to X_x$.

Definition 1.2.1. With the same notations as above, a (G-)cover patching data for the pair (X, Z) consists of the following.

(i) A finite (Galois) cover $Y' \to X'$ (with Galois group G).
(ii) For each point $x \in Z$, a finite (Galois) cover $Y_x \to X_x$ (with Galois group G).
The above data (i) and (ii) must satisfy the following compatibility condition.
(iii) If $\{X_{x,i}\}_{i=1}^n$ are the boundaries of the formal fibre at the point x, then for each $i \in \{1, \ldots, n\}$ is given a (G-equivariant) X_x-isomorphism

$$\sigma_i : Y_x \times_{X_x} X_{x,i} \xrightarrow{\sim} Y' \times_{X'} X_{x,i}.$$

Property (iii) should hold for each $x \in Z$.

The following is the main patching result that we will use in this paper.

Proposition 1.2.2. *With the same notations as above. Given a (G-)cover patching data as in Definition 1.2.1, there exists a unique, up to isomorphism, (Galois) cover $Y \to X$ (with Galois group G) which induces the above (G-)cover in Definition 1.2.1(i), when restricted to X', and induces the above (G-)cover in Definition 1.2.1(ii), when pulled-back to X_x for each point $x \in Z$.*

1.2.3. With the same notations as above, let $x \in Z$ and $\tilde{X}_k$ be the normalisation of X_k. There is a one-to-one correspondence between the set of points of $\tilde{X}_k$ above x, and the set of boundaries of the formal fibre at the point x. Let x_i be the point of $\tilde{X}_k$ above x which corresponds to the boundary $X_{x,i}$, for $i \in \{1, \ldots, n\}$. Assume that the point $x \in X_k(k)$ is rational. Then the completion of $\tilde{X}_k$ at x_i is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ in one variable over k, where t_i is a local parameter at x_i. The complete local ring $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ is a discrete valuation ring with residue field isomorphic to $k((t_i))$. Let $T_i \in \hat{\mathcal{O}}_{x,\mathcal{P}_i}$ be an element which lifts t_i. Such an element is called a parameter of $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$. Then there exists an isomorphism $\hat{\mathcal{O}}_{x,\mathcal{P}_i} \xrightarrow{\sim} R[[T_i]]\{T_i^{-1}\}$ where

$$R[[T]]\{T^{-1}\} \stackrel{\text{def}}{=} \left\{ \sum_{i=-\infty}^{\infty} a_i T^i, \ \lim_{i \to -\infty} |a_i| = 0 \right\}$$

and $|\ |$ is a normalised absolute value of R.

As a direct consequence of the above patching result and the theorems of liftings of étale covers (cf. [3]) one obtains the following (well-known) local-global principle for liftings of (Galois) covers of curves.

Proposition 1.2.4. *Let X be a proper, flat, algebraic (or formal) R-curve and let $Z \stackrel{\text{def}}{=} \{x_i\}_{i=1}^n$ be a finite set of closed points of X. Let $f_k : Y_k \to X_k$ be a finite generically separable (Galois) cover (with Galois group G) with branch locus contained in Z. Assume that for each $i \in \{1, \ldots, n\}$ there exists a (Galois) cover $f_i : Y_i \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x_i}$ (with Galois group G) which lifts the cover $\hat{Y}_{k,x_i} \to \operatorname{Spec} \hat{\mathcal{O}}_{X_k,x_i}$ induced by f_k, where $\hat{\mathcal{O}}_{X_k,x_i}$ (resp. $\hat{Y}_{k,x_i}$) denotes the completion of X_k at x_i (resp. the completion of Y_k above x_i). Then there exists a unique, up to isomorphism, (Galois) cover $f : Y \to X$ (with Galois group G) which lifts the cover f_k and which is isomorphic to the cover f_i when pulled back to $\operatorname{Spf} \hat{\mathcal{O}}_{X,x_i}$, for each $i \in \{1, \ldots, n\}$.*

2. Pro-p quotients of the geometric Galois group of a p-adic open disc

2.1. Notations

The following notations will be used in this section, unless we specify otherwise.

$p > 1$ is a fixed prime integer.
R will denote a complete discrete valuation ring of unequal characteristic, with uniformising parameter π.
$K \stackrel{\text{def}}{=} \operatorname{Fr}(R)$ is the quotient field of R, $\operatorname{char}(K) = 0$.
$k \stackrel{\text{def}}{=} R/\pi R$ is the residue field of R which we assume to be algebraically closed of characteristic $p > 0$.
v_K will denote the valuation of K which is normalised by $v_K(\pi) = 1$.

For an R-(formal) scheme X we will denote by $X_K \stackrel{\text{def}}{=} X \times_R K$ (resp. $X_k \stackrel{\text{def}}{=} X \times_R k$) the generic (resp. special) fibre of X.

2.2

Next, we would like to state a result of Garuti in [2] which concerns the structure of the pro-p geometric fundamental group of a p-adic annulus of thickness zero (cf. Proposition 2.2.3).

First, we recall how one defines the fundamental group of a rigid analytic affinoid space. Let $\mathcal{X} = \mathrm{Spf}\ \mathcal{A}$ be an affine R-formal scheme which is topologically of finite type. Thus, $\mathcal{A}$ is a π-adically complete noetherian R-algebra. Let $A \stackrel{\mathrm{def}}{=} \mathcal{A} \otimes_R K$ be the corresponding Tate algebra and $X \stackrel{\mathrm{def}}{=} \mathrm{Sp}\ A$ the associated rigid analytic affinoid space, which is the generic fibre of $\mathcal{X}$ in the sense of Raynaud (cf. [1]). Assume that X is integral and geometrically connected. Let η be a geometric point of the affine scheme Spec A above the generic point of Spec A. Then η determines naturally an algebraic closure $\overline{K}$ of K and a geometric point of $\mathrm{Spec}(A \times_K \overline{K})$, which we will also denote by η.

Definition 2.2.1 (*Étale Fundamental Groups of Affinoid Spaces*)**.** (See also [2] Définition 2.2 and Définition 2.3.) We define the étale fundamental group of X with base point η by

$$\pi_1(X, \eta) \stackrel{\mathrm{def}}{=} \pi_1(\mathrm{Spec}\ A, \eta),$$

where $\pi_1(\mathrm{Spec}\ A, \eta)$ is the étale fundamental group of the connected scheme Spec A with base point η in the sense of Grothendieck (cf. [3]). Thus, $\pi_1(X, \eta)$ naturally classifies rigid analytic coverings $Y \to X$, where $Y = \mathrm{Sp}\ B$, and B is a finite A-algebra which is étale over A. There exists a natural continuous surjective homomorphism

$$\pi_1(X, \eta) \twoheadrightarrow \mathrm{Gal}(\overline{K}, K).$$

We define the geometric fundamental group $\pi_1(X, \eta)^{\mathrm{geo}}$ of X so that the following sequence is exact:

$$1 \to \pi_1(X, \eta)^{\mathrm{geo}} \to \pi_1(X, \eta) \to \mathrm{Gal}(\overline{K}, K) \to 1.$$

Remark 2.2.2. If L/K is a finite field extension contained in $\overline{K}/K$, and $X_L \stackrel{\mathrm{def}}{=} X \times_K L$ is the affinoid rigid analytic space obtained from X by extending scalars, then we have a natural commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_L, \eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X_L, \eta) & \longrightarrow & \mathrm{Gal}(\overline{K}/L) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X, \eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \longrightarrow & 1
\end{array}$$

where the two right vertical maps are injective homomorphisms and the left vertical map is an isomorphism. The geometric fundamental group $\pi_1(X, \eta)^{\mathrm{geo}}$ is strictly speaking not the fundamental group of a rigid analytic space (since $\overline{K}$ is not complete). It is, however, the projective limit of fundamental groups of rigid affinoid spaces. More precisely, there exists a natural isomorphism

$$\pi_1(X, \eta)^{\mathrm{geo}} \stackrel{\sim}{\to} \varprojlim_{L/K} \pi_1(X \times_K L, \eta)$$

where the limit is taken over all finite extensions L/K contained in $\overline{K}$.

Next, we introduce some notations involved in the statement of Garuti's result. For a finite field extension L/K contained in $\overline{K}/K$ we will denote by

$$D_0 \stackrel{\mathrm{def}}{=} D_{0,L} \stackrel{\mathrm{def}}{=} \mathrm{Sp}\ L < X >$$

the unit closed disc (centred at $X = 0$) and

$$C_L \stackrel{\mathrm{def}}{=} \mathrm{Sp}\ \frac{L < X, Y >}{(XY - 1)}$$

the annulus of thickness 0 which is the "boundary" of D_0. Here $L < X >$ (resp. $L < X, Y >$) denotes the Tate algebra in the variable X (resp. the variables X and Y). We denote by $\mathbb{P}^1_L$ the rigid analytic projective line over L which is obtained by patching the closed discs $D_0 = D_{0,L} \stackrel{\mathrm{def}}{=} \mathrm{Sp}\ L < X >$ and $D_\infty = D_{\infty,L} \stackrel{\mathrm{def}}{=} \mathrm{Sp}\ L < Y >$ along the annulus C_L (see above), via the identification $X \mapsto \frac{1}{Y}$. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite set of closed points of $\mathbb{P}^1_K$ which contains $\{0, \infty\}$ and such that $S \cap C_K = \emptyset$. We view $S \subset \mathbb{P}^1_K$ as a closed subscheme of $\mathbb{P}^1_K$ and write $S_L \stackrel{\mathrm{def}}{=} S \times_K L$. Let η be a geometric point of $\mathbb{P}^1_{\overline{K}} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_K \times_K \overline{K}$ above the generic point of $\mathbb{P}^1_{\overline{K}}$. We denote by $\pi_1(\mathbb{P}^1_L \setminus S_L, \eta)$ the algebraic étale fundamental group of $\mathbb{P}^1_L \setminus S_L$ with base point η. Write

$$C \stackrel{\mathrm{def}}{=} C_K \stackrel{\mathrm{def}}{=} \mathrm{Sp}\ \frac{K < X, Y >}{(XY - 1)}.$$

The natural embedding $C \times_K L = C_L \to \mathbb{P}^1_L$ induces a natural continuous homomorphism

$$\pi_1(C, \eta)^{\mathrm{geo}} \to \pi_1(\mathbb{P}^1_L \setminus S_L, \eta)$$

and by passing to the projective limit a continuous homomorphism

$$\pi_1(C,\eta)^{\mathrm{geo}} \to \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta) \stackrel{\mathrm{def}}{=} \varprojlim_{L/K} \pi_1(\mathbb{P}^1_L \setminus S_L, \eta),$$

where L/K runs over all finite extensions contained in $\overline{K}$. Let $\pi_1(C,\eta)^{\mathrm{geo},p}$ be the maximal pro-p quotient of $\pi_1(C,\eta)^{\mathrm{geo}}$ and $\pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$ the maximal pro-p quotient of $\pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)$. The above homomorphism $\pi_1(C,\eta)^{\mathrm{geo}} \to \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)$ induces a natural continuous homomorphism

$$\phi_S : \pi_1(C,\eta)^{\mathrm{geo},p} \to \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p,$$

which induces by passing to the projective limit a continuous homomorphism

$$\phi \stackrel{\mathrm{def}}{=} \varprojlim_S \phi_S : \pi_1(C,\eta)^{\mathrm{geo},p} \to \varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p,$$

where the limit is taken over all finite set of closed points of $\mathbb{P}^1_K \setminus C$ which contain $\{0,\infty\}$. The profinite pro-p group $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$ is a free pro-p group as follows from the well-known structure of algebraic fundamental groups of affine curves in characteristic 0 (cf. [3]). The following result is one of the main technical results in [2] which we will use in this section (cf. proof of Theorem 2.3.1 and proof of Theorem 2.4.1).

Proposition 2.2.3 (*Garuti*)**.** *The natural continuous homomorphism*

$$\phi \stackrel{\mathrm{def}}{=} \varprojlim_S \phi_S : \pi_1(C,\eta)^{\mathrm{geo},p} \to \varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$$

where the limit is taken over all finite set of closed points of $\mathbb{P}^1_K \setminus C$ which contain $\{0,\infty\}$ makes $\pi_1(C,\eta)^{\mathrm{geo},p}$ into a direct factor of $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S, \eta)^p$. In particular, $\pi_1(C,\eta)^{\mathrm{geo},p}$ is a free pro-p group.

Proof. See [2], Lemma 2.11. □

2.3

Next, we will investigate the structure of a certain quotient of the "geometric absolute Galois group" of a p-adic open disc.

First, we will define this quotient (see the definition of the profinite group Δ below). Write

$$\tilde{X} \stackrel{\mathrm{def}}{=} \operatorname{Spec} R[[T]]$$

and

$$\tilde{X}_K \stackrel{\mathrm{def}}{=} \tilde{X} \times_R K = \operatorname{Spec}(R[[T]] \otimes_R K).$$

The scheme $\tilde{X}_K$ is what we shall refer to as a p-adic open disc (over K). Let $\tilde{S} \stackrel{\mathrm{def}}{=} \{x_1, x_2, \ldots, x_n\} \subset \tilde{X}_K$ be a finite set of closed points of $\tilde{X}_K$. We view $\tilde{S} \subset \tilde{X}_K$ as a closed subscheme of $\tilde{X}_K$. Write

$$U_{K,\tilde{S}} \stackrel{\mathrm{def}}{=} \tilde{X}_K \setminus \tilde{S}$$

and let η be a geometric point of $\tilde{X}_K$ above the generic point of $\tilde{X}_K$. We have a natural exact sequence of profinite groups:

$$1 \to \pi_1(U_{K,\tilde{S}} \times_K \overline{K}, \eta) \to \pi_1(U_{K,\tilde{S}}, \eta) \to \operatorname{Gal}(\overline{K}, K) \to 1.$$

By passing to the projective limit over all finite set of closed points $\tilde{S} \subset \tilde{X}_K$ we obtain a natural exact sequence:

$$1 \to \varprojlim_{\tilde{S}} \pi_1(U_{K,\tilde{S}} \times_K \overline{K}, \eta) \to \varprojlim_{\tilde{S}} \pi_1(U_{K,\tilde{S}}, \eta) \to \operatorname{Gal}(\overline{K}/K) \to 1.$$

Let $L \stackrel{\mathrm{def}}{=} \operatorname{Fr}(R[[T]])$ be the quotient field of the formal power series ring $R[[T]]$. The generic point η determines an algebraic closure $\overline{L}$ of L. We have a natural exact sequence of Galois groups:

$$1 \to \operatorname{Gal}(\overline{L}/\overline{K}.L) \to \operatorname{Gal}(\overline{L}/L) \to \operatorname{Gal}(\overline{K}/K) \to 1.$$

Moreover, there exist natural identifications:

$$\operatorname{Gal}(\overline{L}/\overline{K}.L) \xrightarrow{\sim} \varprojlim_{\tilde{S}} \pi_1(U_{K,\tilde{S}} \times_K \overline{K}, \eta)$$

and

$$\mathrm{Gal}(\bar{L}/L) \xrightarrow{\sim} \varprojlim_{\tilde{S}} \pi_1(U_{K,\tilde{S}}, \eta)$$

where $\tilde{S}$ is as above. The profinite Galois group $\mathrm{Gal}(\bar{L}/\overline{K}.L)$ is what we shall refer to as the geometric Galois group of a p-adic open disc. Let

$$I \stackrel{\text{def}}{=} I_{(\pi)} \subset \mathrm{Gal}(\bar{L}/\overline{K}.L)$$

be the normal closed subgroup which is generated by the inertia subgroups above the ideal (π) of $R[[T]]$. Write

$$\overline{\Delta} \stackrel{\text{def}}{=} \mathrm{Gal}(\bar{L}/\overline{K}.L)/I.$$

Note that, by definition, the profinite group $\overline{\Delta}$ classifies finite Galois covers $\tilde{Y}_{K'} \to \tilde{X}_{K'} \stackrel{\text{def}}{=} \tilde{X} \times_R K'$ where K' is a finite extension of K with valuation ring R', K' is algebraically closed in $\tilde{Y}_{K'}$, π' is a uniformising parameter of K', and the natural morphism $\tilde{Y}' \to \tilde{X}' \stackrel{\text{def}}{=} \tilde{X} \times_R R'$ where $\tilde{Y}'$ is the normalisation of $\tilde{X}'$ in $\tilde{Y}_{K'}$, is étale above the generic point of the special fiber $\tilde{X}'_k \stackrel{\text{def}}{=} \tilde{X}' \times_{R'} k$ of $\tilde{X}'$. In particular, the special fiber $\tilde{Y}'_k \stackrel{\text{def}}{=} \tilde{Y}' \times_{R'} k$ is reduced and the natural morphism $\tilde{Y}'_k \to \tilde{X}'_k$ is generically étale. Let

$$\Delta \stackrel{\text{def}}{=} \overline{\Delta}^p$$

be the maximal pro-p quotient of $\overline{\Delta}$. Our main technical result in this section is the following:

Theorem 2.3.1. *The profinite group Δ is a free pro-p group.*

Proof. The two main ingredients of the proof are the technical result of Garuti in Proposition 2.2.3 and a result of Harbater, Katz, and Gabber (cf. [4,5]). We will show that the profinite pro-p group Δ satisfies property (iii) in Proposition 1.1.1. Let $Q \twoheadrightarrow P$ be a surjective homomorphism between finite p-groups. Let $\phi : \Delta \twoheadrightarrow P$ be a surjective homomorphism. We will show that ϕ lifts to a homomorphism $\psi : \Delta \to Q$. The homomorphism ϕ corresponds to a finite Galois extension $\bar{L}'/\overline{K}.L$ with Galois group P. We can (without loss of generality) assume that this extension is defined over K, thus descends to a finite Galois extension L'/L with Galois group P where K is algebraically closed in L'.

Let A be the integral closure of $R[[T]]$ in L'. We have a finite morphism $f : \mathrm{Spec}\ A \to \mathrm{Spec}\ R[[T]]$ which is (by assumptions) Galois with Galois group P, is étale above the point $(\pi) \in \mathrm{Spec}\ R[[T]]$ and with $\mathrm{Spec}\ A$ geometrically connected. In particular, f induces at the level of special fibres a finite generically Galois cover $\bar{f} : \mathrm{Spec}(A/\pi A) \to \mathrm{Spec}\ k[[t]]$ (where $t = T \mod \pi$) with Galois group P. We will assume (in order to simplify the arguments below) that $\mathrm{Spec}(A/\pi A)$ is connected, the general case is treated in a similar fashion. By a result of Harbater, Katz and Gabber (cf. [4,5]) there exists a finite Galois cover $\bar{g} : \overline{Y} \to \mathbb{P}^1_k$ with Galois group P, which is étale outside a unique closed point ∞ of $\mathbb{P}^1_k$ with local parameter t, $\bar{g}$ is totally ramified above ∞, $\overline{Y}$ is smooth and connected and such that the Galois cover above the formal completion $\mathrm{Spec}\ \hat{\mathcal{O}}_{\mathbb{P}^1_k,\infty}$ of $\mathbb{P}^1_k$ at ∞ which is naturally induced by $\bar{g}$ is generically isomorphic to $\bar{f}$. Let $\mathbb{A}^1_k \stackrel{\text{def}}{=} \mathbb{P}^1_k \setminus \{\infty\}$. The restriction $\bar{f}' : \overline{Y}' \to \mathbb{A}^1_k$ of $\bar{g}$ to $\mathbb{A}^1_k$ is an étale Galois cover with Galois group P and with $\overline{Y}'$ connected.

Consider the rigid analytic projective line $\mathbb{P}^1_K$ which is obtained by patching the closed unit disc $D_\infty \stackrel{\text{def}}{=} \mathrm{Sp}\ K < T >$ (centred at $T = \infty$) with the closed disc $D_0 \stackrel{\text{def}}{=} \mathrm{Sp}\ K < S >$ (centred at $S = 0$) along the annulus $C \stackrel{\text{def}}{=} \mathrm{Sp}\ K < T, S > /(ST - 1)$ (of thickness 0), via the identification $S \mapsto \frac{1}{T}$. The étale Galois cover $\bar{f}'$ lifts (uniquely up to isomorphism) to an étale Galois cover $f' : Y' \to D_0$ by the theorems of liftings of étale covers (cf. [3]), whose restriction $\tilde{f}' : \tilde{Y}' \to C$ to the annulus C is an étale Galois cover with Galois group P. By using formal patching techniques one can construct a (connected) rigid analytic Galois cover $g : Y \to \mathbb{P}^1_K$ with Galois group P whose restriction to the annulus C is isomorphic to $\tilde{f}'$ and which above the formal completion at $T = \infty$ induces the above Galois cover f (see the arguments used in [2], and Proposition 1.2.2). The Galois cover g is ramified above a finite set of closed points $\tilde{S} \subset \mathbb{P}^1_K$ (which are contained in the interior $D^{\text{op}}_\infty \stackrel{\text{def}}{=} D_\infty \setminus C$ of the closed disc D_∞), hence gives rise naturally to a surjective homomorphism $\phi_1 : \pi_1(\mathbb{P}^1_{\overline{K}} \setminus \tilde{S}_{\overline{K}}, \eta) \twoheadrightarrow P$ and also a surjective homomorphism $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p \twoheadrightarrow P$ (where S and the projective limit are as in Proposition 2.2.3). We also denote by $\phi_1 : \pi_1(C, \eta)^{\text{geo},p} \twoheadrightarrow P$ the corresponding homomorphism induced on the direct factor $\pi_1(C, \eta)^{\text{geo},p}$ of $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$.

Next, let $\bar{f}_1 : \mathrm{Spec}\ \overline{B} \to \mathrm{Spec}\ k[[t]]$ be a finite connected Galois cover with Galois group Q, with $\overline{B}$ normal, and which generically dominates the above Galois cover $\bar{f} : \mathrm{Spec}(A/\pi A) \to \mathrm{Spec}\ k[[t]]$ with Galois group P. Note that such $\bar{f}_1$ exists since the maximal pro-p quotient of the absolute Galois group of $k((t))$ is a free pro-p group. Let $\bar{g}_1 : \overline{Y}_1 \to \mathbb{P}^1_k$ be the finite Galois cover with Galois group Q which is étale outside ∞, which induces above $\mathrm{Spec}\ \hat{\mathcal{O}}_{\mathbb{P}^1_k,\infty}$ a finite Galois cover which is isomorphic to $\bar{f}_1$, and let $\bar{g}'_1 : \overline{Y}'_1 \to \mathbb{A}^1_k$ be its restriction to $\mathbb{A}^1_k$ (the Galois cover $\bar{g}_1$ exists by the above result of Harbater, Katz and Gabber (cf. loc. cit.)). By construction the Galois cover $\bar{g}_1 : \overline{Y}_1 \to \mathbb{P}^1_k$ dominates the Galois cover $\bar{g} : \overline{Y} \to \mathbb{P}^1_k$. The étale Galois cover $\bar{g}'_1$ lifts to a finite étale Galois cover $f'_1 : Y'_1 \to D_0$ with Galois group Q, which by construction dominates the lifting $f' : Y' \to D_0$ of $\bar{f}'$. The restriction of f'_1 to the annulus C is a finite étale Galois cover $\tilde{f}'_1 : \tilde{Y}'_1 \to C$ with Galois group Q which dominates the Galois cover $\tilde{f}' : \tilde{Y}' \to C$.

Let N be a complement of $\pi_1(C,\eta)^{\text{geo},p}$ in $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$ (cf. Proposition 2.2.3). The Galois cover $\tilde{f}'_1 : \tilde{Y}'_1 \to C$ (resp. $\tilde{f}' : \tilde{Y}' \to C$) corresponds to the continuous homomorphism $\phi_2 : \pi_1(C,\eta)^{\text{geo}} \to Q$ (resp. $\phi_1 : \pi_1(C,\eta)^{\text{geo},p} \to P$) and ϕ_2 dominates ϕ_1 (by construction). Also the above homomorphism $\phi_1 : \varprojlim \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p \twoheadrightarrow P$ induces naturally a continuous homomorphism $\psi_1 : N \to P$. The pro-p group N being free one can lift the homomorphism ψ_1 to a homomorphism $\psi_2 : N \to Q$ which dominates ψ_1. The profinite pro-p group $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$ being the free direct product of N and $\pi_1(C,\eta)^{\text{geo},p}$ one can construct a continuous homomorphism $\psi : \varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p \to Q$ which restricts to ϕ_2 on the factor $\pi_1(C,\eta)^{\text{geo},p}$ and to ψ_2 on the factor N. Moreover, $\psi : \pi_1(\mathbb{P}^1_{\overline{K}} \setminus (S_0)_{\overline{K}}, \eta)^p \to Q$ factors through $\pi_1(\mathbb{P}^1_{\overline{K}} \setminus (S_0)_{\overline{K}}, \eta)^p$ for some set of closed points $S_0 \subset \mathbb{P}^1_K$ and $S_0 \cap C = \emptyset$. The homomorphism ψ corresponds (after eventually a finite extension of K) to a Galois cover $Y \to \mathbb{P}^1_K$ with Galois group Q which induces naturally a finite Galois cover $g : \text{Spec } B \to \text{Spec } R[[T]]$ with Galois group Q above the formal completion at $T = \infty$, which is by construction étale above the ideal (π) and which dominates the Galois cover $f : \text{Spec } A \to \text{Spec } R[[T]]$ we started with. This in turn corresponds to a homomorphism $\psi : \Delta \to Q$ with the required properties. □

The author doesn't know, and is interested to know, the answer to the following question.

Questions 2.3.2. Is the maximal pro-p quotient $\text{Gal}(\overline{L}/\overline{K}.L)^p$ of the (geometric) Galois group $\text{Gal}(\overline{L}/\overline{K}.L)$ a free pro-p group?

2.4

Next, we investigate a certain quotient of the "geometric absolute Galois group" of the boundary of a p-adic open disc (see the definition of the profinite groups Δ' below).

Let $R[[T]]\{T^{-1}\} \overset{\text{def}}{=} \{\sum_{i=-\infty}^{\infty} a_i T^i,\ \lim_{i\to-\infty} |a_i| = 0\}$ be as in 1.2.3. Note that $R[[T]]\{T^{-1}\}$ is a complete discrete valuation ring with uniformising parameter π and residue field the formal power series field $k((t))$, where $t = T \mod \pi$. Write

$$\mathcal{X} \overset{\text{def}}{=} \text{Spec } R[[T]]\{T^{-1}\}.$$

The scheme $\mathcal{X}$ is what we shall refer to as the boundary of a p-adic open disc (over K). Let

$$M \overset{\text{def}}{=} \text{Fr}(R[[T]]\{T^{-1}\})$$

be the quotient field of the discrete valuation ring $R[[T]]\{T^{-1}\}$. Assume that the generic point η of $\tilde{X}_K$ above (cf. 2.3) arises from a generic point η of $R[[T]]\{T^{-1}\} \otimes_R K$. In particular, the generic point η determines then an algebraic closure $\overline{M}$ of M. We have a natural exact sequence of Galois groups

$$1 \to \text{Gal}(\overline{M}/\overline{K}.M) \to \text{Gal}(\overline{M}/M) \to \text{Gal}(\overline{K}/K) \to 1.$$

Let $I' \overset{\text{def}}{=} I'_{(\pi)} \subset \text{Gal}(\overline{M}/\overline{K}.M)$ be the normal closed subgroup which is generated by the inertia subgroups above the ideal (π) of $R[[T]]\{T^{-1}\}$. Write

$$\overline{\Delta}' \overset{\text{def}}{=} \text{Gal}(\overline{M}/\overline{K}.M)/I'.$$

Note that, by definition, the profinite group $\overline{\Delta}'$ classifies finite Galois covers $\mathcal{Y}_L \to \mathcal{X}_L \overset{\text{def}}{=} \mathcal{X} \times_R L$ where L is a finite extension of K with valuation ring R', L is algebraically closed in $\mathcal{Y}_L$ and the natural morphism $\mathcal{Y} \to \mathcal{X}' \overset{\text{def}}{=} \mathcal{X} \times_R R'$ where $\mathcal{Y}$ is the normalisation of $\mathcal{X}'$ in $\mathcal{Y}_L$ is étale. The natural morphism

$$\text{Spec } R[[T]]\{T^{-1}\} \to \text{Spec } R[[T]]$$

induces a natural homomorphism (cf. 2.3 for the definition of $\overline{\Delta}$ and Δ)

$$\overline{\Delta}' \to \overline{\Delta}.$$

Let

$$\Delta' \overset{\text{def}}{=} \overline{\Delta}'^p$$

be the maximal pro-p quotient of $\overline{\Delta}'$. We have a natural homomorphism

$$\Delta' \to \Delta.$$

Our next technical result in this section is the following.

Theorem 2.4.1. *There exists a natural homomorphism $\Delta' \to \Delta$ (cf. 2.4) which makes Δ' into a direct factor of the free pro-p group Δ. In particular, Δ' is a free pro-p group.*

Proof. One has to verify the cohomological criterion in Proposition 1.1.3 for being a direct factor. Let $f' : \mathcal{Y} \to \mathcal{X}$ be an étale $\mathbb{Z}/p\mathbb{Z}$-torsor. One has to construct (eventually after a finite extension of K) a finite generically Galois cover $f : \tilde{Y} \to \tilde{X} \stackrel{\text{def}}{=} \operatorname{Spec} R[[T]]$ of degree p which induces above $\mathcal{X}$, by pull-back via the natural morphism $\mathcal{X} \stackrel{\text{def}}{=} \operatorname{Spec} R[[T]]\{T^{-1}\} \to \tilde{X} \stackrel{\text{def}}{=} \operatorname{Spec} R[[T]]$, the $\mathbb{Z}/p\mathbb{Z}$-torsor f'.

The torsor f' induces naturally a finite generically Galois cover $\bar{f}' : \mathcal{Y}_k \to \mathcal{X}_k = \operatorname{Spec} k((t))$ of degree p. There exists (as is easily verified, cf. also [5]) a finite Galois cover $\bar{g} : Y_k \to \mathbb{P}^1_k$ of degree p which is ramified above a unique point $\infty \in \mathbb{P}^1_k$, and such that the Galois cover induced by $\bar{g}$ above the formal completion $\operatorname{Spec} \hat{\mathcal{O}}_{\mathbb{P}^1_k,\infty}$ of $\mathbb{P}^1_k$ at ∞ is generically isomorphic to $\bar{f}'$. Let $\bar{g}' : Y'_k \to \mathbb{A}^1_k \stackrel{\text{def}}{=} \mathbb{P}^1_k \setminus \{\infty\}$ be the restriction of $\bar{g}$ which is an étale $\mathbb{Z}/p\mathbb{Z}$-torsor above $\mathbb{A}^1_k$. The étale torsor $\bar{g}'$ lifts (uniquely up to isomorphism) to an étale $\mathbb{Z}/p\mathbb{Z}$-torsor $g' : Y' \to D_0 \stackrel{\text{def}}{=} \operatorname{Sp} K\langle S\rangle$ (where D_0 is the closed disc centred at $S = 0$), by the theorems of liftings of étale covers (cf. [3]), whose restriction $\tilde{g}' : \tilde{Y}' \to C$ to the annulus C is an étale $\mathbb{Z}/p\mathbb{Z}$-torsor which corresponds to a continuous homomorphism $\psi : \pi_1(C, \eta)^{\text{geo},p} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$.

The geometric fundamental group $\pi_1(C, \eta)^{\text{geo},p}$ being a direct factor of $\varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p$ the above homomorphism ψ arises (by restriction) from a continuous homomorphism $\psi' : \varprojlim_S \pi_1(\mathbb{P}^1_{\overline{K}} \setminus S_{\overline{K}}, \eta)^p \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$, and the latter gives rise naturally to a Galois cover $g : Y \to \mathbb{P}^1_K$ of degree p (this cover only exists a priori over a finite extension of K but we can, without loss of generality, assume that it is defined over K) whose restriction to the annulus C is isomorphic (by construction) to the above Galois cover $\tilde{g}'$. The Galois cover g induces naturally a Galois cover above $\tilde{X} \stackrel{\text{def}}{=} \operatorname{Spec} R[[T]]$ (i.e. above the formal completion at $T = \infty$) which induces above the boundary $\mathcal{X}$ the torsor f' as required. $\square$

2.5

In [2] Garuti investigated the problem of lifting of Galois covers between smooth curves. In this sub-section we will prove a refined version of the main result in [2] using Theorems 2.3.1 and 2.4.1. First, we recall the following main result of Garuti.

Theorem 2.5.1 (*Garuti*)**.** *Let X be a proper, smooth and geometrically connected R-curve. Let*

$$f_k : Y_k \to X_k \stackrel{\text{def}}{=} X \times_R k$$

be a finite (possibly ramified) Galois cover between smooth k-curves with Galois group G. Then there exists a finite extension R'/R and a finite morphism

$$f' : Y' \to X' \stackrel{\text{def}}{=} X \times_R R'$$

which is generically étale and Galois with Galois group G, satisfying the following properties:

(i) *Y' is a proper and normal R'-curve.*

(ii) *The natural morphism $f'_k : Y'_k \stackrel{\text{def}}{=} Y' \times_R k \to X'_k = X_k$ is generically étale and Galois with Galois group G. Moreover, there exists a G-equivariant birational morphism $\nu : Y_k \to Y'_k \stackrel{\text{def}}{=} Y' \times_R k$ such that the following diagram is commutative:*

$$\begin{array}{ccc} Y_k & \xrightarrow{\nu} & Y'_k \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f'_k} \\ X_k & \xrightarrow{\mathrm{id}_{X_k}} & X_k \end{array}$$

and the morphism ν is an isomorphism outside the divisor of ramification in the morphism $f_k : Y_k \to X_k$.

(iii) *The special fiber Y'_k is reduced, unibranch, and the morphism $\nu : Y_k \to Y'_k$ is a morphism of normalisation. In particular, Y_k and Y'_k are homeomorphic.*

Proof. (cf. [2], proof of Théorème 2). $\square$

In light of the above result we define Garuti liftings as follows.

Definition 2.5.2 (*Garuti Liftings of Galois Covers between Smooth Curves*)**.** Let X be a proper, smooth and geometrically connected R-curve. Let

$$f_k : Y_k \to X_k \stackrel{\text{def}}{=} X \times_R k$$

be a finite (possibly ramified) Galois cover with Galois group G. Let

$$f' : Y' \to X' \stackrel{\text{def}}{=} X \times_R R'$$

be as in Theorem 2.5.1 for some finite extension R'/R. We call f' a Garuti lifting of the Galois cover f_k (defined over R'). We say that f' is a smooth lifting of f_k if Y' is a smooth R-curve, which is equivalent to the above morphism $\nu : Y_k \to Y'_k$ being an isomorphism. Note that, by definition, a Garuti lifting is defined (a priori) over a finite extension of R. Also, if f_k is étale then a smooth lifting of f_k always exists over R as follows from the theorems of liftings of étale covers (cf. [3]).

The following theorem is a refined version of the above theorem of Garuti.

Theorem 2.5.3. *Let X be a proper, smooth and geometrically connected R-curve. Let*

$$f_k : Y_k \to X_k \overset{\text{def}}{=} X \times_R k$$

be a finite (possibly ramified) Galois cover with Galois group G between smooth k-curves. Assume that the finite group G sits in an exact sequence

$$1 \to H' \to G \to H \to 1.$$

Let

$$Y_k \xrightarrow{g_k} Z_k \xrightarrow{h_k} X_k$$

be the corresponding factorisation of the Galois cover f_k. Thus, $h_k : Z_k \to X_k$ is a finite Galois cover with Galois group H between smooth k-curves. Let

$$h' : Z' \to X' \overset{\text{def}}{=} X \times_R R'$$

be a Garuti lifting of the Galois cover h_k defined over the finite extension R'/R (cf. Definition 2.5.2). Then there exists a finite extension R''/R' and a Garuti lifting

$$f'' : Y'' \to X'' \overset{\text{def}}{=} X \times_R R''$$

of the Galois cover f_k over R'' which dominates h', i.e. we have a factorisation

$$f'' : Y'' \xrightarrow{g''} Z'' \overset{\text{def}}{=} Z \times_{R'} R'' \xrightarrow{h'' \overset{\text{def}}{=} h' \times_{R'} R''} X''$$

where $g'' : Y'' \to Z''$ is a finite morphism between normal R''-curves.

Proof. The proof is (in some sense) similar to the proof of Théorème 2 in [2] using the above Theorem 2.4.1. More precisely, using the techniques of formal patching (cf. [2], and Proposition 1.2.2) the proof of Theorem 2.5.3 follows directly from the following local result in Theorem 2.5.5. □

Before stating our main local result we first define the local analog of Garuti liftings.

Definition 2.5.4 (*Local Garuti Liftings*). Let $\tilde{X} \overset{\text{def}}{=} \operatorname{Spec} R[[T]]$ and $\tilde{X}_k \overset{\text{def}}{=} \operatorname{Spec} k[[t]]$. Let G be a finite group and

$$f_k : \tilde{Y}_k \to \tilde{X}_k$$

a finite morphism which is generically Galois with Galois group G, with $\tilde{Y}_k$ connected and normal. We call a Garuti lifting of the Galois cover f_k, over the finite extension R'/R, a finite Galois cover

$$f' : \tilde{Y}' \to \tilde{X}' \overset{\text{def}}{=} X \times_R R'$$

with Galois group G where R'/R is a finite extension, the morphism $f'_k : \tilde{Y}'_k \to \tilde{X}_k$ is generically Galois with Galois group G, there exists a birational G-equivariant morphism $\nu : \tilde{Y}_k \to \tilde{Y}'_k$ which is a morphism of normalisation and a factorisation

$$f_k : \tilde{Y}_k \xrightarrow{\nu} \tilde{Y}'_k \xrightarrow{f'_k} \tilde{X}_k.$$

The following is our main result which is a refined version of the local version of Garuti's main Theorem 2.5.1.

Theorem 2.5.5. *Let $\tilde{X} \overset{\text{def}}{=} \operatorname{Spec} R[[T]]$ and $\tilde{X}_k \overset{\text{def}}{=} \operatorname{Spec} k[[t]]$. Let G be a finite group and*

$$f_k : \tilde{Y}_k \to \tilde{X}_k$$

a finite morphism which is generically Galois with Galois group G, with $\tilde{Y}_k$ normal and connected. Let H be a quotient of G and

$$h_k : \tilde{Z}_k \to \tilde{X}_k$$

the corresponding Galois sub-cover with Galois group H. Let

$$h' : \tilde{Z}' \to \tilde{X}' \overset{\text{def}}{=} \tilde{X} \times_R R'$$

be a Garuti lifting of h_k over a finite extension R'/R (cf. Definition 2.5.4). Then there exists a finite extension R''/R' and a Garuti lifting

$$f'' : \tilde{Y}'' \to \tilde{X}'' \overset{\text{def}}{=} \tilde{X} \times_R R''$$

of f_k over R'' which dominates h', i.e. we have a factorisation:

$$f'' : \tilde{Y}'' \to \tilde{Z}'' \overset{\text{def}}{=} \tilde{Z}' \times_{R'} R'' \xrightarrow{h'' \overset{\text{def}}{=} h' \times_{R'} R''} \tilde{X}''.$$

Proof. The Galois group G is a solvable group which is a semi-direct product of a cyclic group of order prime-to-p by a p-group. By similar arguments as the ones used by Garuti in [2] it suffices to treat the case where G is a p-group (see the arguments used in [2], Théorème 2.13, and Corollaire 1.11). In this case the proof follows from Theorem 2.4.1. More precisely, assume that G is a p-group (hence H is also a p-group). The Galois cover $f_k : \tilde{Y}_k \to \tilde{X}_k$ is generically given by an étale Galois cover $\operatorname{Spec} k((s)) \to \operatorname{Spec} k((t))$ with Galois group G. This étale cover lifts uniquely to an étale Galois cover $\mathcal{Y} \to \mathcal{X} \overset{\text{def}}{=} \operatorname{Spec} R[[T]]\{T^{-1}\}$ above the boundary of the open disc $\tilde{X}$ with Galois group G and which corresponds to a continuous homomorphism $\psi_2 : \Delta' \to G$ (cf. 2.4 for the definition of Δ').

Let N be a complement of Δ' in Δ (cf. Theorem 2.4.1). The local Garuti lifting $h' : \tilde{Z}' \to \tilde{X}' \overset{\text{def}}{=} \tilde{X} \times_R R'$ corresponds to a continuous homomorphism $\phi : \Delta \to H$ which restricts to continuous homomorphisms $\psi_1 : \Delta' \to H$ and $\phi_1 : N \to H$. The above homomorphism ψ_2 dominates by construction the homomorphism ψ_1. The pro-p group N being free one can lift the homomorphism ϕ_1 to a continuous homomorphism $\phi_2 : N \to G$ which dominates ϕ_1. The pro-p group Δ being isomorphic to the direct free product $\Delta' \star N$, both ψ_2 and ϕ_2 give rise to a continuous homomorphism $\phi' : \Delta \to G$ which dominates the above morphism ϕ. The homomorphism ϕ' in turn corresponds to a Galois cover $\tilde{Y}'' \to \tilde{X}'' \overset{\text{def}}{=} \tilde{X} \times_R R''$ over some finite extension R''/R which is a Garuti lifting of $f_k : \tilde{Y}_k \to \tilde{X}_k$ and which by construction dominates the Garuti lifting $h' : \tilde{Z}' \to \tilde{X}' \overset{\text{def}}{=} \tilde{X} \times_R R'$ of the sub-cover $h_k : \tilde{Z}_k \to \tilde{X}_k$ as required. □

Remark 2.5.6. We assumed in this section that R is of unequal characteristic. In fact the main results of this section: Theorems 2.3.1, 2.4.1, 2.5.3 and 2.5.5 are also valid in the case of a complete discrete valuation ring R of equal characteristic $p > 0$. Indeed, the result of Garuti (cf. Proposition 2.2.3) that we use in the proofs of Theorems 2.3.1 and 2.4.1 is valid in this case (cf. [2]).

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